Exercise 73

A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

Solution

Let t be the time in hours since 7 AM, let $s_1(t)$ be the monk's distance from the monastery on day 1, and let $s_2(t)$ be the monk's distance from the monastery on day 2. The aim is to determine whether there is a value of t that satisfies the equation,

$$s_1(t) = s_2(t).$$

Bring both functions to the left side.

$$s_1(t) - s_2(t) = 0$$

At time $t = 0$, $s_1(t)$ is zero since the monk is at the monastery, and $s_2(t)$ is a very large positive number since the monk is far away from the monastery.

$$\text{At } t = 0, \quad s_1(t) - s_2(t) \text{ is negative.}$$

At time $t = 12$, $s_1(t)$ is a very large positive number since the monk is far away from the monastery, and $s_2(t)$ is zero since the monk is at the monastery.

$$\text{At } t = 12, \quad s_1(t) - s_2(t) \text{ is positive.}$$

The monk's position on both days is continuous on the closed interval $[0, 12]$, and $N = 0$ lies between $s_1(0) - s_2(0)$ and $s_1(12) - s_2(12)$. By the Intermediate Value Theorem, then, there exists a root within $0 < t < 12$. In other words, there is a point on the path that the monk will cross at exactly the same time of day on both days.